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The Multilayer Approximation for Infrasonic Wave Propagation in a Temperature- and Wind-Stratified Atmosphere[1]

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ABSTRACT

The multilayer approximation previously used to study propagation in wind-free atmospheres is extended to include winds. Two generalized acoustic potentials are defined which are continuous even at horizontal discontinuities in wind velocity or sound speed and the residual equations which these quantities satisfy are derived. Two dispersion functions are defined whose roots give the phase-velocity magnitude and phase-velocity direction for normal-mode waves propagating with given frequency and given group-velocity direction. The multilayer approximation is introduced for computing these dispersion functions by approximating continuously stratified atmospheres with models consisting of a finite number of layers, each with constant wind velocity and sound speed. Numerical methods for finding the roots of the dispersion function are discussed. The theory is then applied to an example of a multilayered atmosphere and curves, for several horizontal group-velocity directions, of phase-velocity, group-velocity, and phase-velocity direction versus frequency are tabulated for several normal modes.

INTRODUCTION

The consideration of the propagation of infrasonic waves in the atmosphere generally, particularly at lower frequencies, requires full wave methods. The most practical method for numerical computations of propagation in realistic atmos-

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pheres which has been developed to date is the multilayer method which was devised independently by Pfeffer and Zarichny [1], [2] and by Press and Harkrider [3], [4] for atmospheres without winds. In the application of the multilayer method to wave-propagation problems, the continuously stratified atmosphere is approximated by an atmosphere consisting of a finite number of layers—each layer having constant sound speed. The uppermost layer is taken as unbounded from above. The author [5] has recently proven that numerical solutions to wave-propagation problems computed with this approximation will approach those for any continuously stratified atmosphere, if the number of layers in taken sufficiently large.

Previous applications of the multilayer method have been confined to atmospheres without winds. Studies of the effects of winds on guided low-frequency waves have been made, however, by Weston and VanHulsteyn [6] and by the present author [7]. The former authors studied the propagation of free waves without consideration of the source and derived the eigenvalue problem governing the propagation of guided waves. The present author gave a general formulation of the problem of waves from a point source in a temperature- and wind-stratified atmosphere and showed how the general solution can be expressed at large horizontal distances in a series of terms which can be identified as the analog of the normal-mode waves existing in atmospheres without winds. Numerical results in both of these cited papers were limited, however, due to the lack of a convenient computational method.

In this note, the multilayer approximation is extended to the case where the ambient atmosphere has horizontal winds which vary with height in both direction and magnitude. The ambient temperature of the atmosphere is also assumed to vary with height. This extension is prompted by the author's recent investigations [7] which indicate that atmospheric winds have an appreciable effect on infrasonic wave propagation in the atmosphere. The present paper is confined to the development of the numerical method and its application to the computation of phase and group velocities of normal modes (i.e., guided modes). Other applications (such as the synthesis of transient waveforms from actual sources) will be relegated to later investigations.

In the first section, the basic differential equations for acoustic–gravity wave propagation in a temperature- and wind-stratified atmosphere are derived from the linearized equations of hydrodynamics in a form convenient for the application of the multilayer approximation. Although these residual equations, given by Eqs. (10), differ formally from those derived in a previous paper by the author [7], the equivalence of the two sets of equations can be readily established. The principal advantage of the equations in the form derived here is that none of the coefficients depend explicitly on the derivatives of sound speed or wind speed with respect to height. This implies that the two generalized acoustic "potentials" which satisfy the residual equations will be continuous—even at a surface of discontinuity. The continuity of these potentials is also demonstrated as ensuing from the requirements that total pressure (ambient plus acoustic) and vertical displacement be continuous at horizontal discontinuities up to first order in acoustic amplitudes.

In the second section, a normal-mode dispersion function F is defined. The simultaneous roots of this function and an associated dispersion function P obtainable from F give the two horizontal components of the wave-propagation vector, or, alternatively, the phase velocity and phase-velocity direction, for normal-mode waves of given frequency and group-velocity direction. In the third section, it is shown how the normal-mode dispersion function and the associated dispersion function may be computed by the multilayer method.

Finally, in the fourth section, numerical methods for the computation of phase and group velocity curves are discussed and the theory is applied to an example of a temperature- and wind-stratified atmosphere.

I. Derivation of the Residual Equations

Our starting point for the residual equations is the linearized equations of hydrodynamics for air in the absence of viscosity and thermal conductivity. These, which represent a set of coupled partial differential equations for the deviations p, ϱ, $\mathbf{u}$ of the pressure, density, and fluid velocity from their ambient values p_0, ϱ_0, and $\mathbf{v}$, may be written in the Eulerian form as

$$\varrho_0[D_t\mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{v}] = -\boldsymbol{\nabla} p - g\varrho\mathbf{e}_z, \tag{1a}$$

$$D_t\varrho + \mathbf{u} \cdot \boldsymbol{\nabla}\varrho_0 + \varrho_0\boldsymbol{\nabla} \cdot \mathbf{u} = 0, \tag{1b}$$

$$(D_t p + \mathbf{u} \cdot \boldsymbol{\nabla} p_0) = c^2(D_t\varrho + \mathbf{u} \cdot \boldsymbol{\nabla}\varrho_0), \tag{1c}$$

where we have abbreviated

$$D_t = \partial/\partial t + \mathbf{v} \cdot \boldsymbol{\nabla}, \tag{2}$$

$$c^2 = \gamma p_0/\varrho_0. \tag{3}$$

Here g, $\mathbf{e}_z$, and γ represent the acceleration of gravity, the unit vector in the vertical direction, and the ratio of specific heats of air, respectively. The quantity c is readily recognized as the speed of sound in a homogeneous atmosphere with the neglect of gravity. In writing these equations, we have assumed that the zeroth order, or ambient variables p_0, ϱ_0, and $\mathbf{v}$, are independent of time t and of the horizontal coordinates x and y. Furthermore, the ambient wind velocity $\mathbf{v}$ is taken to be entirely horizontal (i.e., to have no vertical components). The ambient variables may vary arbitrarily with height z, although the height variations of the quantities p_0 and ϱ_0 are related by the hydrostatic equation

$$dp_0/dz = - g\varrho_0.$$

Equations (1), which represent five scalar equations in five unknowns, may be reduced to two coupled partial differential equations in two unknowns. The method chosen for accomplishing this begins with the introduction of two "potentials" Ψ_1 and Ψ_2 by means of the equations

$$u_z = p_0^{-1/2} D_t \Psi_1, \tag{4a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = p_0^{-1/2} D_t \Psi_2, \tag{4b}$$

where u_z denotes the vertical component of $\mathbf{u}$. We may next express the first-order pressure p in terms of Ψ_1 and Ψ_2 by eliminating $D_t\varrho$ from Eqs. (1b) and (1c), and then inserting Eqs. (4) into the resulting expression, obtaining

$$D_t p = - D_t[\varrho_0 p_0^{-1/2}(c^2 \Psi_2 - g\Psi_1)].$$

(The hydrostatic equation is also used in obtaining this expression.) Since Eqs. (4) do not define Ψ_1 and Ψ_2 uniquely, it is clear from the above relation that, in addition, we may require

$$p = - \varrho_0 p_0^{-1/2}(c^2 \Psi_2 - g\Psi_1), \tag{5}$$

which we shall do. In the terminology of electromagnetic theory, Eq. (5) defines our "gauge".

Our next task is the elimination of ϱ, u_x, and u_y from Eqs. (1). One equation may be obtained by operating on both sides of Eqs. (1a) with the horizontal component of the divergence. Doing this, we obtain

$$\varrho_0[D_t(\boldsymbol{\nabla} \cdot \mathbf{u} - \partial u_z/\partial z) + (\partial \mathbf{v}/\partial z) \cdot \boldsymbol{\nabla} u_z] = - \nabla_2^2 p, \tag{6}$$

where ∇_2^2 represents the horizontal Laplacian.

A second equation is obtained by operating on both sides of the z-component of (1a) with D_t and then eliminating $D_t\varrho$ by use of (1b). The resulting equation is

$$\varrho_0 D_t^2 u_z = - D_t(\partial p/\partial z) + g[\mathbf{u} \cdot \nabla \varrho_0 + \varrho_0 \nabla \cdot \mathbf{u}] \tag{7}$$

The next step is to insert Eqs. (4) and (5) into Eqs. (6) and (7) and thereby obtain two coupled partial differential equations for Ψ_1 and Ψ_2. Doing this, with some algebraic manipulation (which we omit for brevity) and extensive use of the hydrostatic equation and Eq. (3), gives two equations, which we may write in matrix form as

$$D_t^2 \left(\frac{\partial}{\partial z}\right) \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \tag{8}$$

where the matrix operator $[Q]$ has the elements

$$Q_{11} = g\nabla_2^2 - (\tfrac{1}{2}\gamma)(g/c^2)D_t^2$$

$$Q_{12} = D_t^2 - c^2\nabla_2^2$$

$$Q_{21} = c^{-2}(D_t^4 + g^2\nabla_2^2)$$

$$Q_{22} = (\tfrac{1}{2}\gamma)(g/c^2)D_t^2 - g\nabla_2^2$$

From the standpoint of the present paper, the chief utility of the partial differential equations (8) is that they yield the desired residual equations in a direct manner. The latter are obtained under the assumption that

$$\Psi_1 = \Phi_1 \exp[- i(\omega t - k_x x - k_y y)], \tag{9a}$$

$$\Psi_2 = \Phi_2 \exp[- i(\omega t - k_x x - k_y y)], \tag{9b}$$

where Φ_1 and Φ_2 are functions of the height z only and where ω, k_x, and k_y are independent of x, y, z, and t. These three constants represent, respectively, the angular frequency and the x- and y-components of the horizontal wave number vector. We can assume expressions of the form (9) to be solutions of Eqs. (8) since the latter set is invariant under translations in x, y, and t. More general solutions of these equations may be formed as Fourier integrals over ω, k_x, and k_y with the expressions (9) forming the integrands. (See, for example, the author's previous formulation [7] of the expression for the field from a point source in a temperature- and wind-stratified atmosphere.)

If we now substitute Eqs. (9) into Eqs. (8) we obtain the desired residual equations, which have the form (in matrix notation)

$$\frac{d}{dz} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}, \tag{10}$$

where

$$A_{11} = g(k^2/\Omega^2) - \gamma g/(2c^2), \tag{11a}$$

$$A_{12} = 1 - c^2k^2/\Omega^2, \tag{11b}$$

$$A_{21} = g^2k^2/(\Omega^2c^2) - \Omega^2/c^2, \tag{11c}$$

$$A_{22} = - A_{11}, \tag{11d}$$

with

$$\Omega = \omega - k_x v_x - k_y v_y, \tag{12}$$

$$k^2 = k_x^2 + k_y^2. \tag{13}$$

The above completes our derivation of the desired residual equations. We should point out, however, that these are not the only possible set of such equations derivable from the linearized equations of hydrodynamics. In this respect, we note that a broad class of such residual equations may be derived by setting

$$\mathbf{\Phi}' = [D]\mathbf{\Phi},$$

where $[D]$ represents any nonsingular two-by-two matrix. For example, the residual equations given in a previous paper [7] would be obtained by letting $D^{11} = ag/c$, $D_{12} = - ac$, $D_{21} = a/c$, and $D_{22} = 0$, where a is any nonzero quantity not depending on z.

Our reason for choosing Eqs. (10) rather than any other pair of equivalent equations is that Eqs. (10) are particularly amenable to numerical computations. The fact that the elements of the matrix $[A]$ do not contain any of the derivatives of c^2 or $\mathbf{v}$ with respect to height implies that Φ_1 and Φ_2 must be continuous with height even in the event that c^2 and $\mathbf{v}$ are discontinuous. This makes the application of the multi-layer method particularly straightforward. Furthermore, the fact that the trace of $[A]$ is zero implies that the determinant of any matrix $[R]$ is 1, if $[R]$ connects the values of Φ_1 and Φ_2 at a given height z_1 to those at height z, via the relation

$$\begin{bmatrix} \Phi_1(z) \\ \Phi_2(z) \end{bmatrix} = [R] \begin{bmatrix} \Phi_1(z_1) \\ \Phi_2(z_1) \end{bmatrix}.$$

This introduces a certain simplification in the formulas derived in Section III.

Some additional remarks may be made at this point concerning our previous comment that the quantities Φ_1 and Φ_2 should be continuous at layer boundaries. If one proceeds from the requirements that total pressure and vertical particle displacement be continuous to first order in acoustic variables at such boundaries, this can be demonstrated in another manner, providing one is careful to take into account the fact that the discontinuity surfaces move under acoustic disturbances causing the apparent ambient variables at either side of the boundary to vary. The first requirement gives, for harmonic disturbances of the form indicated by Eqs. (9), that

$$i(dp_0/dz)(u_z/\Omega) + p$$

be continuous at layer boundaries. However, it follows from Eqs. (1) and the hydrostatic equation that the above quantity is just $- i\gamma p_0 \nabla \cdot \mathbf{u}/\Omega$. Since the ambient pressure is continuous with altitude, the continuity of

$$\Phi_2 = i p_0^{1/2} \nabla \cdot \mathbf{u}/\Omega$$

follows directly. Similarly, the continuity of vertical displacement implies that u_z/Ω be continuous and thus that

$$\Phi_1 = i p_0^{1/2} u_z/\Omega$$

be continuous at any horizontal surface of discontinuity. The manner of derivation given earlier rests on the fact that the residual equations (10) derived for an arbitrary stratified atmosphere do not have any coefficients which are singular at discontinuities.

II. Definition of the Normal-Mode Dispersion Function

For applications to the method of normal modes (i.e., of finding guided wave solutions to the linearized equations of hydrodynamics), it is necessary to find eigensolutions of the residual equations. Appropriate boundary conditions for the method of normal modes are that (1) Φ_1 be zero at the ground, $z = 0$; and (2) the magnitudes of both Φ_1 and Φ_2 vanish with sufficient rapidity as z approaches infinity in such a manner that integrals over Φ_1^2 and Φ_2^2 be bounded as the upper limit goes to infinity.

As has been shown by the author in a previous paper [7], the eigenvalue $k = (k_x^2 + k_y^2)^{1/2}$ for which these conditions are satisfied for given ω and given direction θ_k of the horizontal wavenumber vector will correspond to a pole in the complex k plane if the field of a point source is expressed as an integral over ω, k, and θ_k. If the residue theorem is invoked and the integral over θ_k is performed by the saddle-point method, the field at large distances will appear as a sum over waves which may be considered as a sum over normal mode waves in analogy with the expressions obtained for acoustic waves in simple waveguides. It should be emphasized that these normal mode waves correspond to neither a complete set nor an orthogonal set of eigenfunctions.

The first boundary condition is required in order that the vertical velocity of the air at the ground be zero and is based on the tacit assumption that the ground is rigid. The second boundary condition is required in order that the corresponding amplitude of the normal mode wave be nonzero. To impose the latter without making the numerical computations too complicated, we assume that the atmosphere above some height z_M is isothermal and has constant winds. In this event the coefficients A_{ij} in the residual equations will be constant above z_M. Thus, solutions of Eqs. (10) for $z > z_M$ satisfying the upper boundary condition are easily found if such exist.

The criterion for the existence of solutions satisfying the upper boundary conditions is that the quantity

$$G^2 = (A_{11})^2 + A_{12}A_{21}, \tag{14}$$

computed for the uppermost layer, be greater than zero. If this criterion is satisfied, then the solutions satisfying the upper boundary condition are, for $z > z_M$, of the general form

$$\Phi_1 = -DA_{12}\exp(-G\varrho) \tag{15a}$$

$$\Phi_2 = D(G + A_{11})\exp(-G\varrho) \tag{15b}$$

where $\varrho = z - z_M$ and G is the positive square root of (14). The quantity D is a constant.

To apply the lower boundary condition, let us assume that a matrix $[R]$ has been computed such that, for arbitrary $\Phi_1{}^U$ and $\Phi_2{}^U$ which represent Φ_1 and Φ_2 at $z = z_M$, the values of Φ_1 and Φ_2 at $z = 0$ are given by

$$\begin{bmatrix}\Phi_1{}^0\\ \Phi_2{}^0\end{bmatrix} = \begin{bmatrix}R_{11} & R_{12}\\ R_{21} & R_{22}\end{bmatrix}\begin{bmatrix}\Phi_1{}^U\\ \Phi_2{}^U\end{bmatrix}. \tag{16}$$

(The method for computing $[R]$ is described in the next section.) Then the lower boundary condition implies that

$$R_{11}\Phi_1{}^U + R_{12}\Phi_2{}^U = 0. \tag{17}$$

It follows from Eqs. (15) and (17) that, if both upper and lower boundary conditions are to be satisfied, then one must have $(G^U)^2 > 0$ and

$$F(\omega, k_x, k_y) = 0 \tag{18}$$

where

$$F(\omega, k_x, k_y) = R_{11}A_{12}^U - R_{12}(G^U + A_{11}^U). \tag{19}$$

In the above, the superscript U implies that the values of the superscripted quantities are those appropriate to the region $z > z_M$.

The function F defined by Eq. (19) shall be called the normal-mode dispersion function. Besides being a function of ω, k_x, and k_y, it will also be a function (or functional) of the height profiles of c, v_x, and v_y. However, the dependence on the former quantities is of principal interest since the atmospheric variables will not be changed during the computation of phase and group velocity curves or during the synthesis of transient waveforms.

When G^2 is negative in the uppermost layer, the normal mode dispersion function is not defined. In this event, we shall simply say that F does not exist.

If $F = 0$ for a particular choice of ω, k_x, and k_y, then the corresponding solution of the residual equations will represent a normal-mode wave which has a horizontal phase velocity $\mathbf{v}_p$, with a magnitude

$$v_p = (\omega^2/k^2)^{1/2}. \tag{20}$$

The direction of $\mathbf{v}_p$ will be denoted by the angle θ_k which may be defined by the equations

$$\cos\theta_k = k_x/k, \tag{21a}$$

$$\sin\theta_k = k_y/k, \tag{21b}$$

where k is the positive square root of k^2.

The group velocity $\mathbf{v}_g$ of the normal-mode wave may be computed from the equations

$$v_{gx} = -(\partial F/\partial k_x)/(\partial F/\partial\omega), \tag{22a}$$

$$v_{gy} = -(\partial F/\partial k_y)/(\partial F/\partial\omega). \tag{22b}$$

Its magnitude will be denoted simply v_g, while the angle, reckoned counterclockwise, which $\mathbf{v}_g$ makes with the x axis will be denoted as θ, such that $v_{gx} = v_g \cos\theta$ and $v_{gy} = v_g \sin\theta$.

In this paper, we shall be interested in the computation of phase and group velocities for fixed group-velocity direction θ. This restriction is made for the convenience of comparing computations with microbarovariograph data obtained at a single site. If a particular instrument is recording pressure variations due to normal-mode waves excited by a localized source, then all of the normal-mode waves recorded should have the same group-velocity direction. Thus, the angle θ should be a constant for any given microbarovariograph recording.

If θ is fixed, then the set of values of ω, k_x, and k_y which satisfy Eq. (18) is restricted by the additional relation

$$P(\omega, k_x, k_y, \theta) = 0 \tag{23}$$

where

$$P(\omega, k_x, k_y, \theta) = (\partial F/\partial k_x) \sin\theta - (\partial F/\partial k_y) \cos\theta. \tag{24}$$

This follows from Eqs. (22). Since both $\partial F/\partial k_x$ and $\partial F/\partial k_y$ are functions of ω, k_x, k_y which may be computed, the function P (which we call the auxiliary dispersion function) may be considered as a known function of ω, k_x, and k_y.

For given ω, the two equations (18) and (23) will have a discrete and therefore denumerable set of solutions for k_x and k_y, which we label with the index $n = 1$, 2, 3, etc. The solution pair (k_{xn}, k_{yn}) will be said to correspond to the nth normal mode. The phase velocity v_{pn} and the phase velocity direction θ_{kn} of the nth normal mode can be computed from the Eqs. (20) and (21), while the group-velocity magnitude v_{gn} may be computed from the equation

$$v_{gn} = [(\partial F/\partial k_x)^2 + (\partial F/\partial k_y)^2]^{1/2}/|\partial F/\partial\omega| \tag{25}$$

with $k_x = k_{xn}$ and $k_y = k_{yn}$.

The indicing of the normal modes may be chosen in such a manner that k_{xn}, k_{yn} and therefore v_{pn}, θ_{kn}, and v_{gn} are piecewise-continuous functions of ω. The quantities v_{pn}, θ_{kn}, and v_{gn} are of particular significance and we shall therefore concern ourselves in the remainder of this paper with the development of a numerical procedure for describing the curves of $v_{pn}(\omega)$, $\theta_{kn}(\omega)$, and $v_{gn}(\omega)$ for specified group-velocity direction θ and specified profiles $c(z)$, $v_x(z)$, and $v_y(z)$.

One small subtlety in the theory should be mentioned at this point. Since Eq. (23) is unchanged if θ is replaced by $\theta + \pi$, it may happen that some of the solutions may correspond to a group-velocity direction of $\theta + \pi$ instead of the prespecified θ. One should in fact find all the solutions of Eqs. (18) and (23) and then discard those which correspond to a group-velocity direction of $\theta + \pi$. In practice, however, this complication can be avoided since θ_{kn} and θ will differ by only a small angle for any realistic model atmosphere. This fortunate occurrence is due to the fact that the wind speeds are small compared to the speed of sound in any height region. To take advantage of the small wind speeds, we find it convenient to consider F and P as functions of ω, v_p, and θ_k rather than

ω, k_x, and k_y. Then, if the method used for solving the Eqs. (18) and (23) considers θ as an initial approximation to θ_{kn}, we should expect the solution for θ_{kn} not to differ from θ by too large an amount.

III. Computational Method for the Normal-Mode Dispersion Function

We now discuss the application of the multilayer method to the computation of the normal-mode dispersion function $F(\omega, k_k, k_y)$ and the auxiliary dispersion function $P(\omega, k_x, k_y, \theta)$ defined in the previous section. Our first task in this respect is to supply the details of just how one would obtain the elements of the matrix $[R]$ which appear in Eq. (19).

In order that machine computations of $[R]$ not be unduly lengthly, we restrict ourselves to multilayer atmospheres, where the lower portion ($z < z_M$) is assumed to consist of a finite number M of layers, each with uniform thickness, constant sound speed, and constant wind-velocity components. Let us note that this restriction is in actuality not a severe restriction since any realistic atmosphere can be approximated [5] by a multilayer model with a sufficiently large number of layers. However, one flaw in the method is that there is as yet no criteria developed for the number of layers which are needed to approximate a given atmosphere when a given accuracy is desired in the elements of $[R]$. Nevertheless, some measure of the validity of a computation based on the multilayer approximation may be obtained by simply repeating the calculation using twice as many layers to approximate the actual atmosphere and then comparing the two answers. This sort of check could be applied at any stage in the computation. In what follows, we shall assume that a sufficiently good multilayer atmospheric model has already been chosen.

The layers are numbered with increasing altitude, the height of the top of the Ith layer with respect to the ground being denoted z_I. There are M layers of finite thickness. The $(M + 1)$th layer extends from z_M to ∞, in accordance with the assumptions made in deriving Eqs. (15). Once the sound speed c_I and the wind-velocity components v_{xI}, v_{yI} for the various layers have been assigned values, the coefficients A_{ij} in the residual equations may be computed via Eqs. (11) for any given layer if ω, k_x, and k_y are specified. Thus, in the Ith layer, Eqs. (10) become

$$\frac{d}{dz}\begin{bmatrix}\Phi_1\\ \Phi_2\end{bmatrix} = [A^I]\begin{bmatrix}\Phi_1\\ \Phi_2\end{bmatrix} \tag{26}$$

for $z_{I-1} < z < z_I$.

Since the coefficients in Eqs. (26) are constant, their general solution is readily obtained. In particular, if Φ_1 and Φ_2 are specified at height z_I, then the values of these quantities, at a height z where $z_{I-1} \leq z < z_I$, are given by

$$\begin{bmatrix} \Phi_1(z) \\ \Phi_2(z) \end{bmatrix} = [S] \begin{bmatrix} \Phi_1(z_I) \\ \Phi_2(z_I) \end{bmatrix}. \tag{27}$$

The elements of the matrix $[S]$ are given by

$$S_{ij} = \{CAI(X)\}\delta_{ij} - H\{SAI(X)\}A_{ij} \tag{28}$$

with

$$H = z_I - z, \tag{29}$$

$$X = (A_{11}^2 + A_{12}A_{21})H^2, \tag{30}$$

$$CAI(X) = \cosh[X^{1/2}] \quad \text{if} \quad X > 0 \tag{31a}$$

$$= \cos[(-X)^{1/2}] \quad \text{if} \quad X < 0, \tag{31b}$$

$$SAI(X) = \{\sinh[X^{1/2}]\}/X^{1/2} \quad \text{if} \quad X > 0 \tag{32a}$$

$$= \{\sin[(-X)^{1/2}]\}/(-X)^{1/2} \quad \text{if} \quad X < 0. \tag{32b}$$

For convenience in writing, we have omitted the superscript I from the A_{ij} in Eqs. (28) and (30).

We make use of the functions $CAI(X)$ and $SAI(X)$ defined by Eqs. (31) and (32) rather than using two different formulas for $[S]$ depending on the sign of $A_{11}^2 + A_{12}A_{21}$. Although the definitions given for these functions would seemingly preclude their being analytic at $X = 0$, this is not the case since the power-series expansions about $X = 0$ of (31a) and (32a) are equal, respectively, to those of (31b) and (32b).

We return now to the central problem of developing a method of computing the matrix $[R]$. Equation (26) and the equations which follow it imply that

$$\begin{bmatrix} \Phi_1(z_{I-1}) \\ \Phi_2(z_{I-1}) \end{bmatrix} = [S^I] \begin{bmatrix} \Phi_1(z_I) \\ \Phi_2(z_I) \end{bmatrix}, \tag{33}$$

where $[S^I]$ represents the matrix $[S]$ computed using

$$H = H_I = z_I - z_{I-1}.$$

Since Φ_1 and Φ_2 are continuous at the boundaries z_I separating the layers, we may use Eqs. (33) to build the relation

$$\begin{bmatrix} \Phi_1(0) \\ \Phi_2(0) \end{bmatrix} = [S^{(1)}][S^{(2)}]\cdots[S^{(M)}] \begin{bmatrix} \Phi_1(z_M) \\ \Phi_2(z_M) \end{bmatrix}$$

where matrix multiplication is implied. We accordingly identify

$$[R] = [S^{(1)}][S^{(2)}]\cdots[S^{(M)}] \tag{34}$$

as the desired equation for the matrix $[R]$. This, together with Eq. (19), gives the necessary formulas for computing the normal-mode dispersion function F for given ω, k_x, and k_y and for any given multilayer atmosphere.

For the computation of the auxiliary dispersion function $P(\omega, k_x, k_y, \theta)$, the derivatives $\partial F/\partial k_x$ and $\partial F/\partial k_y$ are needed, while $\partial F/\partial \omega$ is needed, in addition, for the computation of the group velocity using Eq. (22). One could obtain these derivatives by numerical differentiation of the function F, but this would not be too desirable a method since numerical differentiation is often very inaccurate. A preferable method which we exhibit here is the evaluation of the derivatives from an explicit mathematical expression. This method does not involve any [illegible] of giving more reliable results.

Let us note that the method for computing F developed above gives us an explicit, although cumbersome, expression which may be differentiated according to the usual rules of differential calculus. Let q denote any one of the three variables ω, k_x, or k_y. Then, from Eq. (19),

$$\begin{aligned} \partial F/\partial q = {} & (\partial R_{11}/\partial q)A_{12}^U - (\partial R_{12}/\partial q)(G^U + A_{11}^U) \\ & + R_{11}(\partial A_{12}^U/\partial q) - R_{12}(\partial G^U/\partial q + \partial A_{11}^U/\partial q). \end{aligned} \tag{35}$$

The derivatives of the elements of the matrix $[R]$ may be obtained from a consideration of Eq. (34). Using the well-known rules for the differentiation of a matrix product, we obtain

$$[\partial R/\partial q] = \sum_{I=1}^{M} [D^{(I)}][\partial S^{(I)}/\partial q][U^{(I)}] \tag{36}$$

where

$$[D^{(I)}] = [S^{(1)}][S^{(2)}]\cdots[S^{(I-1)}], \tag{37}$$

$$[U^{(I)}] = [S^{(I+1)}][S^{(I+2)}]\cdots[S^{(M)}]. \tag{38}$$

The matrices $[D^{(1)}]$ and $[U^{(M)}]$ are each defined to be the unit matrix.

To obtain the matrices $\partial[S^{(I)}]/\partial q$ we consider Eqs. (28). Differentiating this expression gives, in matrix notation,

$$[\partial S/\partial q] = [\partial S/\partial X]\partial X/\partial q - H\{SAI(X)\}[\partial A/\partial q], \tag{39}$$

where $[\partial S/\partial X]$ is the matrix whose elements are

$$\partial S_{ij}/\partial X = [d[CAI(X)]/dX\}\delta_{ij} - H\{d[SAI(X)]/dX\}A_{ij}. \tag{40}$$

The derivatives of $CAI(X)$ and $SAI(X)$ may be readily shown to be

$$d[CAI(X)]/dX = (\tfrac{1}{2})SAI(X), \tag{41}$$

$$d[SAI(X)]/dX = [CAI(X) - SAI(X)]/(2X). \tag{42}$$

We can also obtain the $\partial X/\partial q$ in terms of the $\partial A_{ij}/\partial q$ from Eq. (30). These derivatives may in turn be found from Eqs. (11)–(13). The same is true for the derivatives of the quantities G^U and the matrix elements A_{ij}^U which appear in Eq. (35). For brevity, we do not explicitly list these derivatives.

Once the derivatives $\partial F/\partial k_x$ and $\partial F/\partial k_y$ are computed by the method outlined above, the auxiliary dispersion function P may be directly computed from Eq. (24).

In applying this method to machine computations, the computation of F and P is relegated to two separate subroutines. Once these subroutines are coded, using the method outlined above, the details of the computation need not be considered explicitly again. Instead, one needs only to call the relevant subroutine when he needs the value of F or P for given ω, k_x, and k_y.

Some simple guidelines can be given for deciding whether or not a given multilayer-model atmosphere is a valid approximation to a continuously stratified atmosphere. These can be established by examining the effect of doubling the number of layers and noting the change in the matrix R. If such a study is carried through analytically for the case of two layers the following criteria may be derived for the thickness of a given layer

$$k^2H^2(\Delta c/c) \ll 1,$$
$$(g^2k^2/\Omega^4)k^2H^2(\Delta c/c) \ll 1,$$
$$(\Omega^2/c^2)H^2(\Delta c/c) \ll 1,$$
$$[g^2k^2/(\Omega^2c^2)]H^2(\Delta c/c) \ll 1,$$

where Δc is the magnitude of the difference in sound speed between the layer in question and an adjacent layer. Analogous conditions involving the difference Δv of wind speed components may be obtained if $\Delta c/c$ in the above formulas is replaced by $k(\Delta v)/\Omega$.

The presence of Ω in the denominator of some of these conditions indicates that the multilayer approximation is invalid for any mode for which the relationship between $\mathbf{k}$ and ω is such that Ω vanishes at any height. For this reason, one should be careful to limit one's calculations to modes having phase velocities greater than the maximum wind speed. Fortunately, it is these modes which may be expected to be of the greatest interest in applications of the method.

For modes where the phase velocity is of the order of the sound speed at the ground, the principal criteria would appear to be that

$$H^3 \ll [c/|dc/dz|]\lambda^2,$$
$$H^3 \ll [c/|dc/dz|]H_s^2.$$

Here λ is the horizontal wavelength, dc/dz is the sound speed gradient, and H_s is the scale height $c^2/(\gamma g)$. Typically, H_s is of the order of 8 km in the lower atmosphere and $c/|dc/dz|$ is never less than 30 km. Thus, for periods in the 1–10-min range, where λ ranges from 20 to 200 km, one may feel fairly confident in his results if the layer thicknesses are less than 5 km in the regions where the sound speed gradients are largest. This, however, is true only for those modes having phase velocities of the order of the speed of sound at the ground.

Once a multilayer model has been adopted the results computed using it may be expected to be most suspect for those modes which either (1) have phase velocities which are comparable to the maximum wind speed, or (2) for which the parameter g^2k^2/ω^4 is much larger than 1, or (3) for modes at higher frequencies, $\omega \gg g/c$, where layer thicknesses become comparable to $(\lambda^2 c/|dc/dz|)^{1/3}$.

IV. The Computation of Phase- and Group-Velocity Curves

For fixed frequency ω, fixed group-velocity direction θ, and a given multilayer atmosphere, the quantities F and P may be considered as functions of the phase velocity v_p and the phase-velocity direction θ_k. We accordingly consider the problem of finding the roots of the equations $F(v_p, \theta_k) = 0$ and $P(v_p, \theta_k) = 0$. To accomplish this, the following method would be appropriate. First, the zeroth approximation for θ_k is taken as θ. To find the zeroth-order approximations for v_p, the function $F(v_p, \theta)$ is scanned by letting v_p run through a sequence of values with regular intervals starting with some minimum value and terminating at some maximum value. The intervals in which F changes sign define the zeroth-order approximations to the v_{pn}.

Once the zeroth-order approximations for v_p and θ_k are determined, the actual values may be determined by successively solving

$$F(v_{\mathrm{p}}^{(n+1)},\ \theta_k^{(n)}) = 0 \tag{43}$$

for $v_{\mathrm{p}}^{(n+1)}$ and then solving

$$P(v_{\mathrm{p}}^{(n+1)}, \theta_k^{(n+1)}) = 0 \tag{44}$$

for $\theta_k^{(n+1)}$. Here $v_{\mathrm{p}}^{(n)}$ and $\theta_k^{(n)}$ denote the nth order approximations to the roots. Equations (43) and (44) are solved by the Newton–Ralphson method [8] using the values of $v_{\mathrm{p}}^{(n)}$ or $\theta_k^{(n)}$, respectively, for initial approximations to $v_{\mathrm{p}}^{(n+1)}$ and $\theta_k^{(n+1)}$ in the iteration. In practice, the convergence of the method is very rapid for finding the roots to accuracies of 10^{-5} km/sec in either component of the phase velocity.

Once the roots for v_{p} and θ_k are determined, the corresponding values of the group velocity magnitude v_{g} can be computed from Eqs. (25).

This process can be repeated for a sequence of values of ω and thus the curves of $v_{\mathrm{p}n}$, θ_{kn}, and $v_{\mathrm{g}n}$ may be obtained. A somewhat more efficient method, however, would be to carry out this process for a small number of widely spaced values of ω and then to fill in the gaps by following each normal mode separately, letting ω increase in steps of a small increment Δ, using

$$\theta_{kn}(\omega + \Delta) = \theta_{kn}(\omega),$$
$$v_{\mathrm{p}n}(\omega + \Delta) = v_{\mathrm{p}}(\omega) + (\Delta/\omega)[v_{\mathrm{g}}(\omega) - v_{\mathrm{p}}(\omega)]/v_{\mathrm{p}}(\omega)$$

to obtain the zeroth-order approximations for θ_{kn} and $v_{\mathrm{p}n}$ at $\omega + \Delta$ from the values of θ_{kn}, $v_{\mathrm{p}n}$, and $v_{\mathrm{g}n}$ computed at ω.

In applying the multilayer approximation to the calculation of phase- and group-velocity curces, some considerations of machine time should be taken into account. In single precision (eight significant figures), computation of the normal-mode dispersion function on the IBM 7094 requires approximately 2 msec per layer. The simultaneous calculation of the three derivatives of the normal-mode dispersion function requires about 10 msec per layer. If one takes an atmosphere of 50 layers and does a computation which requires 1000 values of the normal-mode dispersion function, a time of about one minute will be expended. Since it is the roots of the normal-mode dispersion function which are desired, it is reasonable to expect that at least five separate computations of the normal-mode dispersion function will be required to find a single root to five significant figures. If one wants simultaneous roots of Eqs. (18) and (24), the number may be much larger. Thus, if the problem is not coded with some care, times comparable to 0.1 min may be required to find a single simultaneous root. If a large number of computations are required, this can lead to a prohibitive expense.

A manner of circumventing this, which we have tried with some success, is to take the zeroth-order approximation for v_p (computed letting $\theta_k = \theta$) as an acceptable solution for v_p, assuming that the deviation $\theta_k - \theta$ is small. Once this approximate value of v_p is found, the value of θ_k is computed from the approximate formula

$$\theta_k = \theta + P/(\partial P/\partial \theta),$$

where θ_k is set equal to θ in the arguments of $P(\omega, v_p, \theta_k, \theta)$ and of $\partial P(\omega, v_p, \theta_k, \theta)/\partial\theta$. The group velocity is then computed by Eq. (22) setting $\theta_k = \theta$. This method of calculation is clearly quite permissible if the winds are sufficiently weak. The drawback of the method is that it does not guarantee the accuracy of v_p and θ_k for nonzero winds. Its primary advantage over the more exact method described previously is that it offers a considerable saving (as much as a factor of five) in computation time. We have coded the problem by both methods and find that, for realistic atmospheres and for the modes of interest which travel with velocities of the order of the speed of sound at the ground, the simple approximation described above is adequate.

To illustrate the effectiveness of the general method, we reproduce here the results of computations carried out for a representative atmosphere of 22 layers. The parameters characterizing this model atmosphere are listed in Table I. The sound-speed profile is typical of midlatitudes and has the characteristic two minimums. The winds are assumed to be entirely in the x direction. This particular wind profile was chosen primarily for the purpose of investigating the effects of stratospheric winds on the wave propagation. The peak wind speed of 72.1 meters/sec (roughly one fourth the speed of sound) is representative [9] of wind speeds which might be expected in this altitude range at midlatitudes.

Figure 1 shows the results of one type of calculation which is particularly suitable for a digital computer. The sign of the normal-mode dispersion function is printed out for an array of discrete values of the phase velocity and the angular frequency for fixed phase-velocity direction. A plus is printed when the function is positive and a minus is printed when it is negative. If it does not exist, an X is printed. The resulting printout gives a good qualitative picture of the lines along which the function is zero. Figure 1 was computed for a phase-velocity direction of 180°. Since the winds are only in the x direction, the group-velocity direction is also 180°. The phase velocities, which represent the vertical coordinate of the figure, range from 0.200 to 0.500 km/sec. The horizontal coordinate is the angular frequency ω which ranges from 0.01 to 0.1 rad/sec with scale divisions of 0.0018 rad/sec. In Figure 1, one may recognize portions of the dispersion curves of 13 modes. Eleven of these have similar forms—the phase velocity for each being

TABLE I

REPRESENTATIVE MODEL OF A MULTILAYER ATMOSPHERE USED IN NUMERICAL COMPUTATIONS

Height of layer bottom (km)	Height of layer top (km)	Speed of sound (km/sec)	Wind velocity (x-component) (km/sec)
0.0	2.0	0.3362	0.0
2.0	4.0	0.3284	0.0
4.0	6.0	0.3204	0.0
6.0	8.0	0.3121	0.0
8.0	10.0	0.3037	0.0
10.0	11.0	0.2972	0.0
11.0	25.1	0.2949	0.0
25.1	30.0	0.3007	0.0051
30.0	34.0	0.3124	0.0154
34.0	38.0	0.3162	0.0309
38.0	42.0	0.3236	0.0412
42.0	36.0	0.3309	0.0618
46.0	53.4	0.3368	0.0669
53.4	60.0	0.3300	0.0721
60.0	70.0	0.3020	0.0618
70.0	80.0	0.2712	0.0309
80.0	91.0	0.2579	0.0154
91.0	108.0	0.2889	0.0
108.0	120.0	0.3830	0.0
120.0	130.0	0.4814	0.0
130.0	140.0	0.5558	0.0
140.0	infinite	0.5667	0.0

greater than 0.5 km/sec at low frequencies, decreasing at first rapidly with increasing frequency, and then decreasing at a slower rate as the frequency increases further. The remaining two modes have lower cutoff frequencies of approximately 0.30 and 0.22 km/sec, respectively.

We have done a calculation similar to that represented by Fig. 1 for the downwind case, where the group and phase velocity directions are both 0°, and we find qualitatively similar results with one interesting exception. An additional mode appears in the lower left-hand corner of the figure. Just how this mode evolves as the direction of propagation is varied is illustrated in Fig. 2. This, which was obtained using the same programing technique as Fig. 1, shows how the phase velocity of the various modes vary with the phase-velocity direction when the frequency is held constant. The vertical coordinate in the figure is the phase velocity and the horizontal coordinate is the phase-velocity direction, which ranges from 0° to 360° in units of 10°. The angular frequency was fixed at 0.012 rad/sec

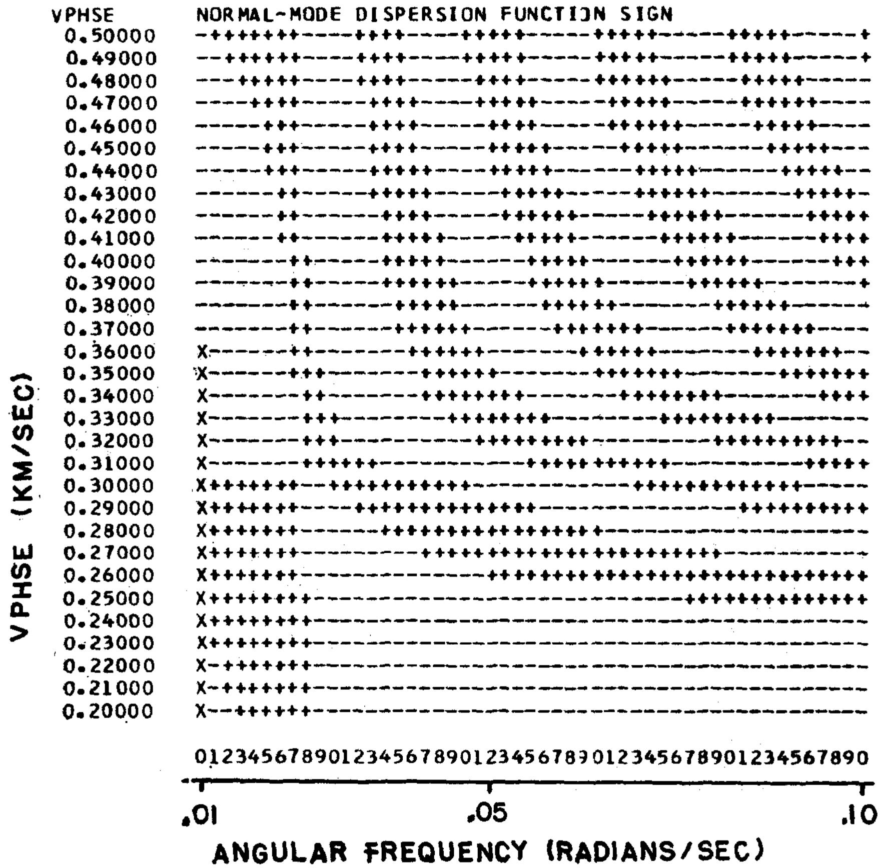

FIG. 1. Digital computer printout of the sign of the normal-mode dispersion function for phase velocities between 0.2 and 0.5 km/sec and angular frequencies between .01 and 0.1 rad/sec. The printout is at angular frequency intervals of .0018 rad/sec and phase-velocity intervals of .01 km/sec. The phase- and group-velocity directions are both 180°.

during the computtion. Anisotropic effects are particularly evident for the modes which have lower phase velocities.

In Table II we list phase and group velocities versus angular frequency for group-velocity directions of 0°, 90°, and 180°, respectively, for the "fundamental mode" shown in Figs. 1 and 2. (By fundamental mode, we mean that mode which has a phase velocity of approximately 0.32 km/sec at very low frequencies in the downwind direction.) The phase-velocity direction is also tabulated for the case of 90° group-velocity direction. It is not tabulated for the cases when the group-velocity direction is 0° or 180° since the phase-velocity direction is automatically

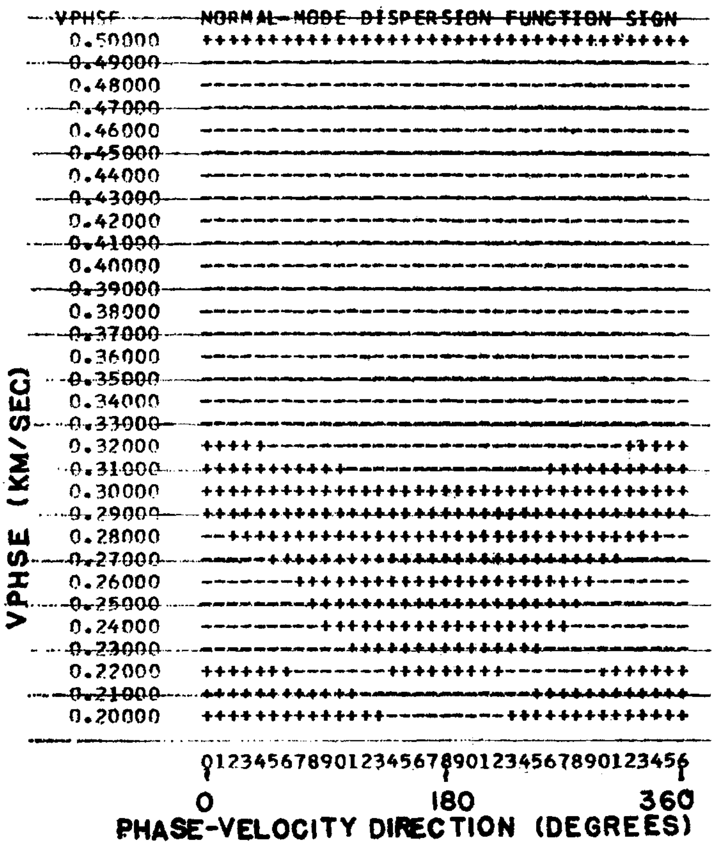

FIG. 2. Digital computer printout of the sign of the normal-mode dispersion function for phase velocities between 0.2 and 0.5 km/sec and phase-velocity directions between 0° and 360°. The printout is at phase-velocity direction intervals of 10° and phase-velocity intervals of 0.01 km/sec. The angular frequency is 0.012 rad/sec.

the same as the group-velocity direction for these cases. Similar computations are presented for the first and second acoustic modes in Tables III and IV, respectively. These modes correspond to the first two curves in Fig. 1 which appear with high phase velocities when the frequency is low. This method of presenting the com-

TABLE II

TABULATION OF PHASE VELOCITY ν_p, GROUP VELOCITY ν_g, AND PHASE-VELOCITY DIRECTION θ_k VERSUS ANGULAR FREQUENCY ω FOR FIXED GROUP-VELOCITY DIRECTION θ FOR THE FUNDAMENTAL MODE

ω (rad/sec)	ν_p (km/sec)			ν_g (km/sec)			θ_k (degrees)
	$\theta° = 0$	$\theta = 90°$	$\theta = 180°$	$\theta = 0°$	$\theta = 90°$	$\theta = 180°$	$\theta = 90°$
.011	.3291	.3119	.3072	.3233	.3117	.3060	91.38
.012	.3286	.3118	.3071	.3218	.3116	.3056	91.38
.013	.3280	.3118	.3070	.3203	.3115	.3051	91.38
.014	.3274	.3118	.3068	.3188	.3114	.3045	91.38
.015	.3267	.3117	.3066	.3173	.3113	.3038	91.38
.016	.3261	.3117	.3064	.3158	.3112	.3028	91.38
.017	.3254	.3117	.3062	.3144	.3110	.3016	91.38
.018	.3247	.3116	.3059	.3130	.3107	.3006	91.39
.019	.3241	.3116	.3055	.3118	.3104	.2976	91.39
.020	.3234	.3115	.3050	.3105	.3098	.2936	91.40
.021	.3227	.3114	.3042	.3091	.3087	.2856	91.41
.022	.3220	.3112	.3028	.3069	.3056	.2619	91.45
.023	.3212	.3106	.2975	.2982	.2876	.1501	91.58
.024	.3157	.2985	.2685	.1073	.0715	.0602	94.40
.025	.2724	.2501	.2289	.0491	.0421	.0433	95.23
.026	.2227	.2043	.1926	.0340	.0330	.0359	94.16

putational results was chosen in preference to plotted curves in order to allow the reader as much detail as possible in quantitatively assessing the effects of winds.

Although our primary purpose here is to demonstrate the feasibility of the computation, several features of the curves tabulated in Tables II, III, and IV should be pointed out. The phase velocity in any given mode for any given frequency is always higher downwind than upwind, although this is not necessarily the case for the group velocity. Differences between the phase- and group-velocity directions may be as large as 16° for propagation crosswind, but this difference varies markedly from mode to mode and with frequency. The general shape of the group-velocity curves does not vary too markedly for the fundamental mode where in all cases it decreases monotonically with increasing frequency. However, the dispersion (i.e., the derivative of the group velocitiy) is apparently less in the crosswind direction than either upwind or downwind. This is in disagreement with the author's previous approximate calculations [7].

TABLE III

Tabulation of Phase Velocity ν_p, group Velocity ν_g, and Phase-Velocity Direction θ_k Versus Angular Frequency ω for Fixed Group-Velocity Direction θ for the First Acoustic Mode

ω (rad/sec)	ν_p (km/sec)			ν_g (km/sec)			θ_k (degrees)
	$\theta = 0°$	$\theta = 90°$	$\theta = 180°$	$\theta = 0°$	$\theta = 90°$	$\theta = 180°$	$\theta = 90°$
.010	.5050	.5050	.5050	.4718	.4717	.4717	90.0
.011	.5012	.5012	.5011	.4600	.4599	.4598	90.0
.012	.4968	.4968	.4968	.4464	.4463	.4462	90.0
.013	.4918	.4918	.4917	.4309	.4307	.4305	90.0
.014	.4861	.4860	.4860	.4132	.4129	.4125	90.0
.015	.4795	.4794	.4793	.3930	.3925	.3919	90.0
.016	.4719	.4718	.4716	.3700	.3692	.3682	90.0
.017	.4632	.6429	.4627	.3439	.3425	.3406	90.0
.018	.4529	.4525	.4520	.3142	.3120	.3084	90.1
.019	·4408	.4401	.4390	.2807	.2769	.2702	90.1
.020	.4262	.4249	.4226	.2432	.2369	.2235	90.2
.021	.4084	.4059	.4002	.2020	.1919	.1659	90.5
.022	.3860	.3810	.3667	.1585	.1436	.1068	90.1
.023	.3574	.3475	.4238	.1160	.0988	.0923	92.3
.024	.3248	.3145	.3179	.1428	.1810	.2214	91.5
.025	.3211	.3121	.3050	.2990	.2987	.2633	91.3
.026	.3203	.3118	.3032	.3055	.3066	.2644	91.3
.027	.3198	.3116	.3014	.3066	.3082	.2548	91.4
.028	.3193	.3115	.2991	.3070	.3087	.2401	91.4
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
.030	.3185	.3113	.2927	.3072	.3087	.2150	91.4
.031	.3181	.3112	.2892	.3072	.3085	.2104	91.5
.032	.3177	.3111	.2858	.3072	.3081	.2096	91.5
.033	.3174	.3110	.2827	.3072	.3077	.2106	91.5
.034	.3171	.3109	.2799	.3072	.3070	.2125	91.6
.035	.3168	.3107	.2775	.3072	.3061	.2145	91.7
.036	.3165	.3105	.2753	.3072	.3047	.2166	91.9
.037	.3163	.3103	.2733	.3072	.3026	.2185	92.1
.038	.3160	.3100	.2716	.3072	.2989	.2202	92.5
.039	.3158	.3094	.2700	.3072	.2917	.2217	93.2
.040	.3156	.3082	.2685	.3072	.2783	.2231	94.6

TABLE IV

TABULATION OF PHASE VELOCITY v_p, GROUP VELOCITY v_g, AND PHASE-VELOCITY DIRECTION θ_k VERSUS ANGULAR FREQUENCY ω FOR FIXED GROUP VELOCITY DIRECTION θ FOR THE SECOND ACOUSTIC MODE

ω (rad/sec)	v_p (km/sec)			v_g (km/sec)			θ_k (degrees)
	$\theta = 0°$	$\theta = 90°$	$\theta = 180°$	$\theta = 0°$	$\theta = 90°$	$\theta = 180°$	$\theta = 90°$
.024	—	—	.4264	—	—	.1092	—
.026	—	.4362	.3620	—	.1342	.1421	106.0
.028	.5148	.3878	.3314	.1491	.1660	.1791	102.2
.030	.4512	.3606	.3182	.1812	.1892	.2316	100.4
.032	.4164	.3438	.3129	.2034	.2072	.2668	99.2
.034	.3938	.3325	.3104	.2173	.2209	.2811	98.2
.036	.3775	.3246	.3088	.2258	.2320	.2857	97.1
.038	.3650	.3191	.3075	.2311	.2432	.2858	95.9
.040	.3550	.3154	.3063	.2351	.2579	.2829	94.a
.042	.3467	.3133	.3050	.2386	.2777	.2775	92.3
.044	.3399	.3121	.3034	.2417	.2941	.2697	91.3
.046	.3341	.3115	.3015	.2447	.3014	.2603	90.9
.048	.3292	.3112	.2993	.2475	.3039	.2512	90.8
.050	.3250	.3109	.2968	.2500	.3049	.2444	90.8
.052	.3214	.3107	.2942	.2523	.3051	.2400	90.8
.054	.3182	.3104	.2917	.2544	.4051	.2376	90.8
.056	.3154	.3103	.2893	.2563	.3049	.2362	90.8

For the first acoustic mode, wind effects are virtually negligible at low frequencies—indicating that the energy is predominantly carried in a layer above the assumed stratospheric winds. Note that a sharp group-velocity minimum is reached for all directions of propagation at an angular frequency of about 0.023 rad/sec. For propagation upwind a group-velocity maximum is encountered at $\omega = 0.024$ rad/sec which is not encountered for propagation downwind and crosswind.

For the second acoustic mode, the group velocity initially increases with increasing frequency. A group-velocity maximum appears in the tabulation at $\omega = .038$ and at $\omega = .053$ for propagation upwind and crosswind, respectively. No such maximum appears for propagation downwind.

The extent to which these features depend on the detailed structure of the atmosphere will be discussed in a later article.

V. Concluding Remarks

Before discussing the principal implications of our analysis, we would like to point out one incidental result which may be of some theoretical interest. This is that the quantities u_z/Ω and $\nabla \cdot \mathbf{u}/\Omega$ are continuous at surfaces at which the horizontal wind velocity and the sound speed (or temperature) are discontinuous. This follows from Eqs. (4), (10), and (11), and the fact that the ambient pressure is continuous with height. Although the continuity of these quantities may be known to some investigators, we doubt that it is widely known. The continuity of the latter quantity certainly does not appear to be intuitively obvious.

The theory presented in this paper has shown that the multilayer approximation may be extended to include winds. Furthermore, it has been demonstrated that the use of the approximation affords a practical method of computing dispersion curves of normal modes in temperature- and wind-stratified atmospheres. Dispersion curves computed by the method should be useful in the interpretation of microbarovariograph data recorded at large distances from localized sources of finite time duration such as volcano eruptions and nuclear tests.

This, however, is but one application of the multilayer approximation. Other applications which remain to be explored for temperature- and wind-stratified atmospheres are the synthesis of actual waveforms from localized sources, the computation of amplitude-height profiles for propagating normal modes, and the development of a theory of ionospheric disturbances generated by acoustic-gravity waves created by weather disturbances. The theory presented in this paper should provide the framework for these applications.

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